RESEARCH ARTICLE

Minimum Covering Maximum Degree Energy of a Graph

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ABSTRACT: In this paper, we introduce the minimum covering Maximum degree energy of a graph. We compute minimum covering Maximum degree energy of some standard graphs and obtain upper and lower bound for this energy. Also obtain the minimum covering Maximum degree energy of thorn graph of some standard graphs.

KEYWORDS: Minimum covering set, minimum covering maximum degree eigenvalues, thorn graph of a graph, minimum covering maximum degree energy.

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1. INTRODUCTION

Graph spectra are a well developed subject of mathematics, where linear algebra and matrix theory are used to study graphs. Using spectra of graphs, different characterizations can be studied. More details about graph spectra are given in [1,2]. There are numerous applications of graph spectra, few are as follows;

Physics: The theory of graph spectra is used in treating the vibration of membrane which arises in chemical physics.

Chemistry: The theory of graph spectra is used in chemistry, in treating unsaturated hydrocarbons by an approximating technique called the Huckle molecular orbital theory.

Computer Science: Graph Spectra determines the expansion properties of a communication network.

Let G be a simple, finite connected graph of order n with vertex set $V=\{x_1, x_2, x_3,\ldots,x_n\}$. The adjacency matrix of G, $A(G)=(a_{ij})$ where $a_{ij}=1$, if x_i and x_j are adjacent, otherwise 0. The eigenvalues of the matrix $A(G)$ are determined by the characteristic equation $|A-\lambda I|=0$. Since $A(G)$ is real and symmetric, the eigenvalues are denoted and labeling in non-increasing order as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$. The energy $E(G)$ of G is defined by

$$E(G)=\sum_{i=1}^{n}|\lambda_i| \quad where \quad i=1,2,\ldots n.$$

The energy of graph was defined by Gutman [3]. In theoretical chemistry, the π-electron energy of a conjugated carbon molecule, computed using Huckle theory, coincides with the energy as defined above. Hence results on graph energy assume special importance in graph theory. More results on graph energy are available in [4-6]. Due to the ample significances of graph energy, many researchers are defined various energy with respect to a graph. Adiga and Smitha defined Maximum degree matrix $M(G)$ of a graph G [7] as follows

$$M(G)=\begin{cases}\max[d(x_i),d(x_j)] & if \;\; x_ix_j \in E \\ 0 & otherwise\,.\end{cases}$$

2. *The Minimum Covering Maximum Degree Energy of a Graph*

Let G be a simple graph of order n with vertex set $V=\{x_1, x_2, x_3,\ldots,x_n\}$ and edge set E. A subset C of V is called a covering set of G, if every edge of G is incident to atleast one vertex of C. Any covering set with minimum cardinality is called minimum covering set. Let C be the minimum covering set of a graph G. The minimum covering maximum degree matrix $A_c[M(G)]=(a_{ij})$ is given by

$$a_{ij}=\begin{cases}\max[d(x_i),d(x_j)] & if \;\; x_ix_j \in E \\ 1 & if \;\; i=j \; and \; if \;\; x_i \in C \\ 0 & otherwise\,.\end{cases}$$

The characteristic polynomial of $A_c[M(G)]$ is denoted by $f_M^C(G,\lambda)=\det(\lambda I - A_c(M(G))$. Since the minimum covering maximum degree matrix is real

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and symmetric. Hence its eigen values are also real and symmetric $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ (with multiplicities $n_1, n_2, n_3, \ldots n_r$). By arranging into non-increasing order we can write $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$. We shall denote the spectrum of $A_c[M(G)]$ as

$$Spec(A_c[M(G)]) = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \cdots \lambda_n \\ n_1 & n_2 & n_3 \cdots n_r \end{pmatrix}.$$

The minimum covering maximum degree energy of G is defined analogous to adjacency energy [3] by

$$E_{CM}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Minimum covering matrix, energy and more results are given in [8]. In this paper we compute various bounds for Minimum Covering Maximum degree energy and energy for some graphs. Also we obtain Minimum Covering Maximum degree energy for thorn graph of a graph.

3. Some Basic Properties of Minimum Covering Maximum Degree Energy of a Graph

Proposition 3.1. *Let G be a graph with vertex set V and edge set E and minimum covering set C. Let* $f_M^C(G, \lambda) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n$. *Then*

$(i) c_0 = 1$

$(ii) c_1 = -|C|$

$(iii) c_2 = |C| - K$

Where $|C|$ *is the trace,* $K = \sum_{i<j} d_i d_j$ *and* $d_i d_j$ *is the product of degrees of two adjacent vertices.*

Proof: (*i*) From the definition of $f_M^C(G, \lambda)$ it follows that $c_0 = 1$.

(*ii*) Since the sum of diagonal elements of $A_c[M(G)]$ is equal to $|C|$, the sum of determinants of all 1×1 principal sub matrices of $A_c[M(G)]$ is the trace of $A_c[M(G)]$ which is obviously equal to $|C|$.

Hence $(-1)^1 = |C|$. That is $c_1 = -|C|$.

(*iii*) $(-1)^2 c_2$ is equal to the sum of determinants of all 2×2 principle sub matrices of $A_c[M(G)]$, that is

$$c_2 = \sum_{1 \leq i < j \leq n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$$

$$= \sum_{1 \leq i < j \leq n} (a_{ii}a_{jj} - a_{ij}a_{ji}) = \sum_{1 \leq i < j \leq n} a_{ii}a_{jj} - \sum_{1 \leq i < j \leq n} a_{ij}^2 = |C| - K.$$

Proposition 3.2. If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ *are the eigenvalues of* $A_c[M(G)]$, *then*

$(i) \sum_{i=1}^{n} \lambda_i = |C|.$

$(ii) \sum_{i=1}^{n} \lambda_i^2 = 2K + |C|$.

Proof: (*i*) $\sum_{i=1}^{n} \lambda_i$ = trace of $A_c[M(G)]$.

= Sum of the diagonal elements.

= $|C|$.

$$(ii)\ \sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji}$$

$$= 2\sum_{i<j}\left(a_{ij}\right)^2 + \sum_{i=1}^{n}(a_{ii})^2$$

$$= 2K + |C|.$$

Theorem 3.3. (Upper bound). *Let G be a graph with n vertices, m edges and let C be a minimum covering set of G. Then*

$$E_{CM}(G) \le \sqrt{n\left(2K + |C|\right)}.$$

Proof: Let $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ be the eigenvalues of $A_c[M(G)]$. By the Cauchy-Schwartz inequality we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Let $a_i = 1$ and $b_i = |\lambda_i|$. Then

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \le \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

$$\le n\sum_{i=1}^{n} \lambda_i^2$$

$$\left(E_{CM}(G)\right)^2 \le n\left(2K + |C|\right),$$

hence we have

$$E_{CM}(G) \le \sqrt{n\left(2K + |C|\right)}.$$

Theorem 3.4. (Lower bound). *Let G be a graph with n vertices, m edges and let C be a minimum covering set of G. If H=*$\left|\det A_c[M(G)]\right|$*, then*

$$E_{CM}(G) \ge \sqrt{2K + |C| + n(n-1)H^{2/n}}\,.$$

Proof: Since we have

$$\left(E_{CM}(G)\right)^2 = \sum_{i=1}^{n} |\lambda_i|^2$$

$$= \left(\sum_{i=1}^{n} |\lambda_i|\right)\left(\sum_{j=1}^{n} |\lambda_j|\right)$$

$$= \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i \ne j} |\lambda_i|\,|\lambda_j|.$$

Employing the inequality between the arithmetic and geometric means we obtain

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \geq \prod_{i\neq j}|\lambda_i||\lambda_j|$$

$$\begin{aligned}(E_{CM}(G))^2 &\geq \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{1/n(n-1)}\\ &\geq \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\left(\prod_{i=1}^{n}|\lambda_i|^{2(n-1)}\right)^{1/n(n-1)}\\ &= \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\left|\prod_{i=1}^{n}\lambda_i\right|^{2/n}\\ &= 2K + |C| + n(n-1)H^{2/n}.\end{aligned}$$

Hence we have

$$E_{CM}(G) \geq \sqrt{2K + |C| + n(n-1)H^{2/n}}.$$

4. Minimum Covering Maximum Degree Energy of Some Standard Graphs

In this section, we obtain the minimum covering maximum degree energy of some standard graphs like, complete graph, star graph, complete bipartite graph, crown graph, cocktail party graph and double star graph.

Theorem 4.1. *The minimum covering maximum degree energy of the complete graph* K_n *is*

$$E_{CM}(K_n) = (n-2)^2 + \sqrt{n^4 - 2n^3 + 3n^2 - 6n + 5}$$

Proof: Let K_n be the complete graph with vertex set $V = \{x_1, x_2, x_3, \ldots, x_n\}$. The minimum covering set is then $C = \{x_1, x_2, x_3, \ldots, x_{n-1}\}$. The minimum covering maximum degree matrix is

$$A_c[M(K_n)] = \begin{bmatrix} 1 & n-1 & n-1 & \cdots & n-1 & n-1\\ n-1 & 1 & n-1 & \cdots & n-1 & n-1\\ n-1 & n-1 & 1 & \cdots & n-1 & n-1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ n-1 & n-1 & \cdots & n-1 & 1 & n-1\\ n-1 & n-1 & \cdots & n-1 & n-1 & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$f_M^C(K_n,\lambda) = (\lambda + n - 2)^{n-2}[\lambda^2 - (n^2 - 3n + 3)\lambda - (n^3 - 3n^2 + 3n - 1)].$$

Hence we have

$$E_{CM}(K_n) = (n-2)^2 + \sqrt{n^4 - 2n^3 + 3n^2 - 6n + 5}.$$

Theorem 4.2. *The minimum covering maximum degree energy of the star graph* $K_{1,n-1}$ *is*

$$E_{CM}(K_{1,n-1}) = (n-1)(n-2) + \sqrt{n^4 - 2n^3 - n^2 + 6n - 3}.$$

Proof: Let $K_{1,n-1}$ be the star graph with vertex set $V=\{x_0,x_1,x_2,\ldots,x_{n-1}\}$ with assumption that x_0 is the central vertex. Then the minimum covering set is $C=\{x_0\}$. The minimum covering maximum degree matrix is

$$A_c[M(K_{1,n-1})]=\begin{bmatrix} 1 & n-1 & n-1 & \cdots & n-1 & n-1 \\ n-1 & 0 & n-1 & \cdots & n-1 & n-1 \\ n-1 & n-1 & 0 & \cdots & n-1 & n-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n-1 & \cdots & n-1 & 0 & n-1 \\ n-1 & n-1 & \cdots & n-1 & n-1 & 0 \end{bmatrix}.$$

The characteristic polynomial will become

$$f_M^C(K_{1,n-1},\lambda)=(\lambda+n-1)^{n-2}\left[\lambda^2-(n^2-3n+3)\lambda-(n^3-4n^2+6n-3)\right].$$

Hence we have

$$E_{CM}(K_{1,n-1})=(n-1)(n-2)+\sqrt{n^4-2n^3-n^2+6n-3}.$$

Theorem 4.3. *The minimum covering maximum degree energy of the complete bipartite graph $K_{n,n}$ is*

$$E_{CM}(K_{n,n})=n-1+\sqrt{1+4n^4}\,.$$

Proof: Let $K_{n,n}$ be the complete bipartite graph of order $2n$ with vertex set

$$V=\{x_1,x_2,x_3,\ldots,x_n,y_1,y_2,y_3,\ldots,y_n\}\,.$$

Here minimum covering set would be $C=\{x_1,x_2,x_3,\ldots,x_n\}$. The minimum covering maximum degree matrix is

$$A_c[M(K_{n,n})]=\begin{bmatrix} 1 & 0 & 0 & \cdots & n & n & n \\ 0 & 1 & 0 & \cdots & n & n & n \\ 0 & 0 & 1 & \cdots & n & n & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n & n & n & \cdots & 0 & 0 & 0 \\ n & n & n & \cdots & 0 & 0 & 0 \\ n & n & n & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial will become

$$f_M^C(K_{n,n}\,,\lambda)=\lambda^{n-1}(\lambda-1)^{n-1}\left[\lambda^2-\lambda-n^4\right].$$

Hence we have

$$E_{CM}(K_{n,n})=n-1+\sqrt{1+4n^4}.$$

Theorem 4.4. *The minimum covering maximum degree energy of the crown graph S_n^0 is*

$$E_{CM}(S_n^0)=(n-1)\sqrt{4n^2-8n+5}+\sqrt{4n^4-16n^3+24n^2-16n+5}\,.$$

Proof: Let S_n^0 be the crown graph of order $2n$ with vertex set

$$V=\{x_1,x_2,x_3,\ldots,x_n,y_1,y_2,y_3,\ldots,y_n\}$$

Here minimum covering set would be $C=\{x_1,x_2,x_3,\ldots,x_n\}$. The minimum covering maximum degree matrix is

$$A_c[M(S_n^0)] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & n-1 & n-1 & \cdots & n-1 \\ 0 & 1 & 0 & \cdots & 0 & n-1 & 0 & n-1 & \cdots & n-1 \\ 0 & 0 & 1 & \cdots & 0 & n-1 & n-1 & 0 & \cdots & n-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & n-1 & n-1 & \cdots & n-1 & 0 \\ 0 & n-1 & n-1 & \cdots & n-1 & 0 & 0 & 0 & \cdots & 0 \\ n-1 & 0 & n-1 & \cdots & n-1 & 0 & 0 & 0 & \cdots & 0 \\ n-1 & n-1 & 0 & \cdots & n-1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ n-1 & n-1 & \cdots & n-1 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The characteristic polynomial will become

$$f_M^C(S_n^0, \lambda) = \left[\lambda^2 - \lambda - (n-1)^2\right]^{n-1} \left[\lambda^2 - \lambda - (n-1)^4\right].$$

Hence we have

$$E_{CM}(S_n^0) = (n-1)\sqrt{4n^2 - 8n + 5} + \sqrt{4n^4 - 16n^3 + 24n^2 - 16n + 5}.$$

Theorem 4.5. *The minimum covering maximum degree energy of the cocktail party graph* $K_{n\times 2}$ *is*

$$E_{CM}(K_{n\times 2}) = 4n^2 - 12n + 9 + \sqrt{16n^4 - 32n^3 + 24n^2 - 24n + 17}.$$

Proof: Let $K_{n\times 2}$ be the cocktail party graph of order $2n$ having vertex set as

$$V = \{x_1, x_2, x_3, \ldots, x_n, y_1, y_2, y_3, \ldots, y_n\}.$$

It is $2n$-2 regular graph and assuming minimum covering set as

$$C = \{x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \ldots, y_{n-1}\}.$$

The minimum covering maximum degree matrix is

$$A_c[M(K_{n\times 2})] = \begin{bmatrix} 1 & 0 & 2n-2 & 2n-2 & 2n-2 & 2n-2 & \cdots & 2n-2 & 2n-2 \\ 0 & 1 & 2n-2 & 2n-2 & 2n-2 & 2n-2 & \cdots & 2n-2 & 2n-2 \\ 2n-2 & 2n-2 & 1 & 0 & 2n-2 & 2n-2 & \cdots & 2n-2 & 2n-2 \\ 2n-2 & 2n-2 & 0 & 1 & 2n-2 & 2n-2 & \cdots & 2n-2 & 2n-2 \\ 2n-2 & 2n-2 & 2n-2 & 2n-2 & 1 & 0 & \cdots & 2n-2 & 2n-2 \\ 2n-2 & 2n-2 & 2n-2 & 2n-2 & 0 & 1 & \cdots & 2n-2 & 2n-2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2n-2 & 2n-2 & 2n-2 & 2n-2 & 2n-2 & 2n-2 & \cdots & 1 & 0 \\ 2n-2 & 2n-2 & 2n-2 & 2n-2 & 2n-2 & 2n-2 & \cdots & 0 & 1 \end{bmatrix}.$$

The characteristic polynomial will become

$$f_M^C(K_{n\times 2}, \lambda) = \lambda(\lambda - 1)^{n-1}(\lambda + 4n - 5)^{n-2}\left[\lambda^2 - (4n^2 - 12n + 9)\lambda - 2(2n-2)^3\right].$$

Hence we have

$$E_{CM}(K_{n\times 2}) = 4n^2 - 12n + 9 + \sqrt{16n^4 - 32n^3 + 24n^2 - 24n + 17}.$$

Theorem 4.6. *The minimum covering maximum degree energy of the double star graph* $B_{m,n}(m>n)$ *is*

$$E_{CM}(B_{m,n}) = |\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4|.$$

Where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are roots of the equation

$$\lambda^4 - 2\lambda^3 - \overline{(m+1)^3 + n(n+1)^2 - 1}\ \lambda^2 + \overline{m(m+1)^2 + n(n+1)^2}\ \lambda + mn(m+1)^2(n+1)^2.$$

Proof: Double star graph $B_{m,n}(m>n)$ obtained from K_2 by joining m pendent edges to one end and n pendent edges to the other end of K_2. Let $B_{m,n}(m>n)$ be the double star graph of order $m+n+2$ with vertex set

$$V = \{x_1, x_2, \ldots, x_m, x_{m+1}, y_1, y_2, \ldots, y_n, y_{n+1}\}.$$

By assuming x_{m+1} and y_{n+1} as end vertices of K_2, the minimum covering set would be $C = \{x_{m+1}, y_{n+1}\}$.
The minimum covering maximum degree matrix is

$$A_c[M(B_{m,n})] = \begin{bmatrix} 1 & m+1 & m+1 & m+1 & \cdots & m+1 & 0 & 0 & \cdots & 0 \\ m+1 & 1 & 0 & 0 & \cdots & 0 & n+1 & n+1 & \cdots & n+1 \\ m+1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ m+1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & n+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & n+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & n+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial will become

$$f_M^C(B_{m,n}, \lambda) = \lambda^{m+n-2}\left[\lambda^4 - 2\lambda^3 - \overline{(m+1)^3 + n(n+1)^2 - 1}\ \lambda^2 + \overline{m(m+1)^2 + n(n+1)^2}\ \lambda + mn(m+1)^2(n+1)^2\right].$$

Hence we have

$$E_{CM}(B_{m,n}) = |\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4|.$$

Where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are roots of the equation

$$\lambda^4 - 2\lambda^3 - \overline{(m+1)^3 + n(n+1)^2 - 1}\ \lambda^2 + \overline{m(m+1)^2 + n(n+1)^2}\ \lambda + mn(m+1)^2(n+1)^2.$$

5. Minimum Covering Maximum Degree Energy of a Thorny Graph

The *thorny graph* G^{+k} is a graph which obtained from G by attaching k pendant vertices to every vertex of G. Thorny graphs are discussed in [9-12].

In this section for r-regular graph G, we obtain a relation between minimum covering maximum degree polynomial of matrix $G^{+k}(k>1)$ and the adjacency polynomial of graph G. Further we compute the above mentioned energy for thorn graph of, complete graph, complete bipartite graph, crown graph and cocktail party graph.

Theorem 5.1. {Relation between characteristic polynomial of $A_c[M(G^{+k})]$ and polynomial of $A(G)$ }.

The minimum covering maximum degree polynomial of $G^{+k}(k>1)$ for r-regular graph G is

$$f_M^C(G,\lambda)=\lambda^{nk}(r+k)^n\left|\left(\frac{\lambda^2-\lambda-k(r+k)^2}{\lambda(r+k)}\right)I-A(G)\right|.$$

Where $A(G)$ is adjacency matrix of G.

Proof: Let G be connected r-regular graph of order n with vertex set $V=\{x_1,x_2,x_3,\ldots,x_n\}$. If $k>1$ the minimum covering set of G^{+k} is simply the vertex set of G. The minimum covering maximum degree matrix would be in the form

$$A_c[M(G^{+k})]=\begin{bmatrix} O_{nk\times nk} & P_{nk\times n} \\ P^T_{n\times nk} & A^*(G)_{n\times n}\end{bmatrix}.$$

Where $A^*(G)=I_n+(r+k)A(G)$ and P is a matrix with i^{th} column C_i having $(r+k)$'s in $[(i-1)k+1]^{th}$ to ik^{th} positions and rest 0's. Where $i=1,2,\ldots,n$.

The characteristic polynomial is

$$f_M^C(G^{+k},\lambda)=\begin{vmatrix} \lambda I_{nk} & -P_{nk\times n} \\ -P^T_{n\times nk} & \lambda I_n-A^*(G)_{n\times n}\end{vmatrix}.$$

Since $P^TP=k(r+k)^2I_n$ and using Schur's lemma $\begin{vmatrix} M & N \\ P & Q\end{vmatrix}=|M|\,|Q-PM^{-1}N|$,

$$f_M^C(G^{+k},\lambda)=\left|\lambda I_n-I_n-(r+k)A(G)-\frac{k(r+k)^2}{\lambda}I_n\right|,$$

$$=\lambda^{nk}(r+k)^n\left|\left(\frac{\lambda^2-\lambda-k(r+k)^2}{\lambda(r+k)}\right)I-A(G)\right|.$$

Hence from the above theorem it is clear that by replacing λ by $\dfrac{\lambda^2-\lambda-k(r+k)^2}{\lambda(r+k)}$ in the characteristic polynomial of $A(G)$ with pre-multiplied by $\lambda^{nk}(r+k)^n$, we can obtain characteristic polynomial of $A_c[M(G^{+k})]$.

Corollary 5.2. *The minimum covering maximum degree energy of $(K_n)^{+k}$ is*

$$E_{CM}(K_n)^{+k}=\sqrt{[(n-1)(r+k)+1]^2+4k(r+k)^2}+(n-1)\sqrt{(1-r-k)^2+4k(r+k)^2}$$

Proof: Let K_n be a complete graph of order n. For K_n the characteristic polynomial is

$$(\lambda-n+1)(\lambda+1)^{n-1}.$$

From theorem 5.1 we have

$$f_M^C[(K_n)^{+k},\lambda]=\lambda^{nk}(r+k)^n\left(\frac{\lambda^2-\lambda-k(r+k)^2}{\lambda(r+k)}-(n-1)\right)\left(\frac{\lambda^2-\lambda-k(r+k)^2}{\lambda(r+k)}+1\right)^{n-1}.$$

On simplifying

$$f_M^C((K_n)^{+k},\lambda)=\lambda^{nk-n}\left[\lambda^2-\overline{(n-1)(r+k)+1}\lambda-k(r+k)^2\right]\left[\lambda^2-\overline{1-r-k}\lambda-k(r+k)^2\right]^{n-1}.$$

Hence we obtain

$$E_{CM}(K_n)^{+k} = \sqrt{[(n-1)(r+k)+1]^2 + 4k(r+k)^2} + (n-1)\sqrt{(1-r-k)^2 + 4k(r+k)^2}\,.$$

Corollary5.3. *The minimum covering maximum degree energy of* $(K_{n,n})^{+k}$ *is*

$$E_{CM}(K_{n,n})^{+k} = (2n-2)\sqrt{1+4k(r+k)^2} + \sqrt{[1-n(r+k)]^2 + 4k(r+k)^2} + \sqrt{[1+n(r+k)]^2 + 4k(r+k)^2}\,.$$

Proof: On similar lines.

Corollary5.4. *The minimum covering maximum degree energy of* $\left(S_n^0\right)^{+k}$ *is*

$$E_{CM}(S_n^0)^{+k} = (n-1)\sqrt{(1-r-k)^2 + 4k(r+k)^2} + \sqrt{(1+r+k)^2 + 4k(r+k)^2} + \sqrt{[1-(n-1)(r+k)]^2 + 4k(r+k)^2} + \sqrt{[1+(n-1)(r+k)]^2 + 4k(r+k)^2}\,.$$

Proof: On similar lines.

Corollary5.5. *The minimum covering maximum degree energy of* $(K_{n\times 2})^{+k}$ *is*

$$E_{CM}(K_{n\times 2})^{+k} = n\sqrt{1+4k(r+k)^2} + (n-1)\sqrt{[1-2(r+k)]^2 + 4k(r+k)^2} + \sqrt{[1+(2n-2)(r+k)]^2 + 4k(r+k)^2}\,.$$

Proof: On similar lines.

CONCLUSION

In this paper we introduced minimum covering maximum degree matrix of a graph and energy and obtained some basic results with bounds for minimum covering maximum degree energy. The relation between adjacency polynomial of a graph *G* and minimum covering maximum degree polynomial of thorn graph of *G* were discussed.

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